

Lecture 4 Portfolio Selection and Mean-Variance Analysis

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Reading

- Textbook: chapter 5
- Bodie, Kane and Marcus, Investments, McGraw Hill, 8th. 2009, chapter 6 &7
- Bodie, Z., R. Merton, D. Cleeton, Financial Economics, Prentice Hall, 2nd edition, 2008, chapter 10, 11.

Outline

- Portfolio Choice under Expected Utility
- Combining riskless asset and a single risky asset
- Portfolios of two risky assets
- Optimal combination of risky assets
- Achieving a target expected return
- Portfolios of many risky assets
- Efficient frontier

I. The portfolio selection problem

- Securities have uncertain return outcomes.
How can we decide which of them to hold?
Called the “portfolio selection problem.”
- Modern portfolio theory (MPT) began with Markowitz (1952)
- Note: Before the development of MPT, investors would often talk about risk and return, but the failure to quantify these two concepts made construction of a portfolio of assets highly subjective.

I. The portfolio selection problem

- The importance of this theory was shown in 1990, when the Nobel Prize was jointly awarded to Prof. Harry Markowitz and Prof. William Sharpe, one of the developers of capital market theory.
- An investor chooses a portfolio that combines the assets in a variety of proportions (known as portfolio weights).
- Generally, there are constrictions on these proportions. Example of a portfolio possibilities set for two restrictions:
 1. All portfolio weights add up to one.
 2. No short sales allowed.

I. The portfolio selection problem

Expected Utility Criterion

- *‘An investor will select the portfolio that yields the highest possible expected value for his or her utility function.’*

A. Portfolio choice problem

- **Portfolio choice problem** can be stated as: choose the portfolio of assets to **maximize expected utility** subject to the **wealth constraint**.
- Terminal wealth is written as

$$W = (1 + r_p) W_0$$

where W_0 is initial wealth, r_p is the portfolio return .

- Thus, the investor's portfolio choice problem is:

$$\begin{aligned} \max EU(W) \\ \text{st. } W = (1 + r_p) W_0 \end{aligned}$$

Example 1

- Suppose that there is one riskless asset and one risky asset.
- The return on the risky asset is specified as follows: 0% with probability p and 20% with probability $1-p$.
- On the other hand, the risk free rate is 5%.
- Assume that you are an expected utility maximizer and your utility function is $u(x)=\exp(cx)$ with $c>0$.
- You now have an initial wealth of W_0 , and by allocating W_0 between the assets, you are trying to maximize the expected utility arising from the wealth at the end of the investment, i.e., a is the investment in the risky asset, and $(W_0 - a)$ is the investment in the riskless asset.

- Now answer the following questions:
 - Set up your optimization problem.
 - Derive the first order condition of your optimization problem.
 - Show that the demand for the risky asset a^* does not depend on W_0 .
 - Show that the demand for the risky asset a^* is decreasing in c .

Summarize:

- The set-up the problem
 - Objective Function
 - Control Variables
 - Constraints
 - Parameters
 - ✓ Utility Function & Probability
 - ✓ Initial Wealth
 - ✓ Prices
- The Solution to the Problem
 - Demand for the assets: optimal portfolio

Remark

- Every portfolio that maximizes expected utility must satisfy a condition called the *fundamental valuation relationship* (FVR)

$$E[(1 + r_j)H] = 1, \text{ for every asset } j.$$

- What is H ? Depends on the context (i.e. assumption about the expected utility function).
- At optimum, increment for expected utility must be the same for every asset.

B. Mean-variance objective

- Each investor who acts according to a mean-variance objective is assumed to choose a portfolio that maximizes

$$U = U(\mu_p, \sigma_p^2) = \mu_p - 0.5A\sigma_p^2$$

subject to the constraint that the total value of assets (calculated at initial prices) does not exceed initial wealth.

B. Mean-variance objective

- Agents thus have 2 conflicting objectives:
 1. Maximize expected return
 2. Minimize risk
- These 2 must be balanced against each other. Risk averse investors are willing to forgo some expected return in exchange for less risk
- This trade off leads to ...
- The **Indifference curves**, i.e. a set of expected return and risk combinations that provide an agent with the same utility.

B. Mean-variance objective

- A mean-variance investor will find any portfolio that is “farther north-west” provides greater utility than any portfolio lying on an indifference curve below it.
- An investor has an infinite number of indifference curves.
- Every investor has a unique indifference map representing preferences.

How can these be determined?

- A mean-variance investor's optimal portfolio selection can be split logically into two steps:
- First, the **portfolio frontier**, comprising those portfolios for which σ_p^2 is minimized for each m_p .
- Second, a choice is made from among the frontier portfolios so as to **maximize** the objective, U , in accordance with preference.

Further:

- Portfolio optimization (frontier) is often done as a two-step process:
 - (1) find the optimal combination of risky assets, and
 - (2) mix this optimal risky asset portfolio with the riskless asset.

II. Combining the riskless asset and a single risky asset

- The expected return of the portfolio is the weighted average of the component returns

$$\mu_p = \omega_1^* \mu_1 + \omega_2^* \mu_2$$

$$\mu_p = \omega_1^* \mu_1 + (1 - \omega_1)^* \mu_2$$

- The volatility of the portfolio is :

$$\sigma_p = ((\omega_1^* \sigma_1)^2 + 2\omega_1 \omega_2 \rho \sigma_1 \sigma_2 + (\omega_2^* \sigma_2)^2)^{1/2}$$

- We know something special about the portfolio, namely that security 2 is riskless, so $\sigma_2 = 0$, and σ_p becomes:

$$\sigma_p = ((\omega_1 * \sigma_1)^2 + 2\rho\omega_1 * \sigma_1 * \omega_2 * 0 + (\omega_2 * 0)^2)^{1/2}$$

$$\sigma_p = |\omega_1| * \sigma_1$$

– In summary

$$\sigma_p = |\omega_1| * \sigma_1, \text{ And:}$$

$$\mu_p = \omega_1 * \mu_1 + (1 - \omega_1) * r_f, \text{ So:}$$

$$\text{If } \omega_1 > 0, \mu_p = [(\mu_1 - r_f) / \sigma_1] * \sigma_p + r_f$$

$$\text{Else } \omega_1 < 0, \mu_p = [(r_f - \mu_1) / \sigma_1] * \sigma_p + r_f$$

- Risk-return trade-off line

$$\mu_p = [(\mu_T - r_f) / \sigma_T] * \sigma_p + r_f$$

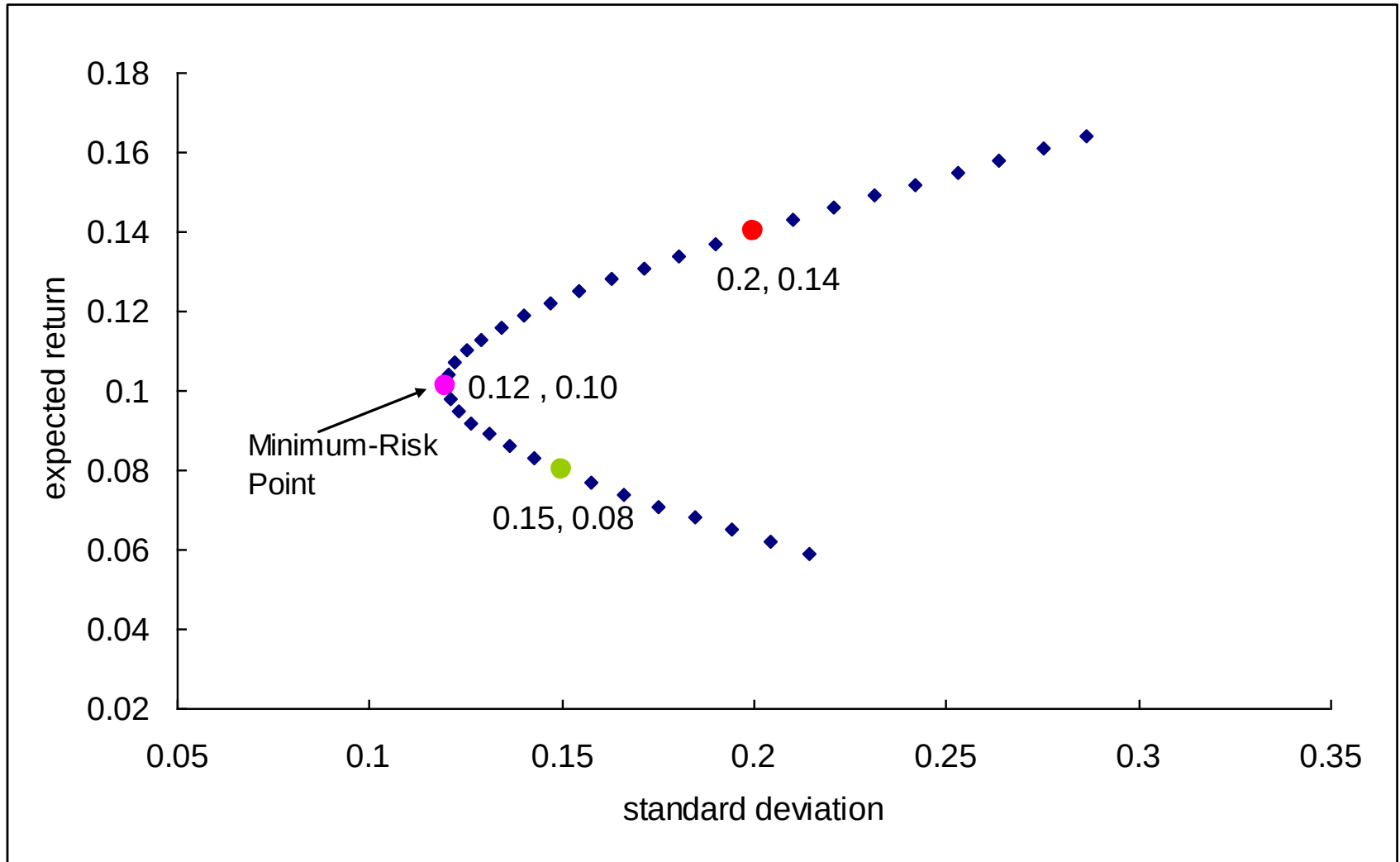
Example 2

Distribution of rates of return on two risky assets

	Risky asset 1	Risky asset 2
Mean	.14	.08
Standard deviation	.20	.15
Correlation	0	0

- Find the weight ω that minimize the variance of the portfolio (**Minimum risk portfolio**, MRP).

III. Portfolio of two risky assets

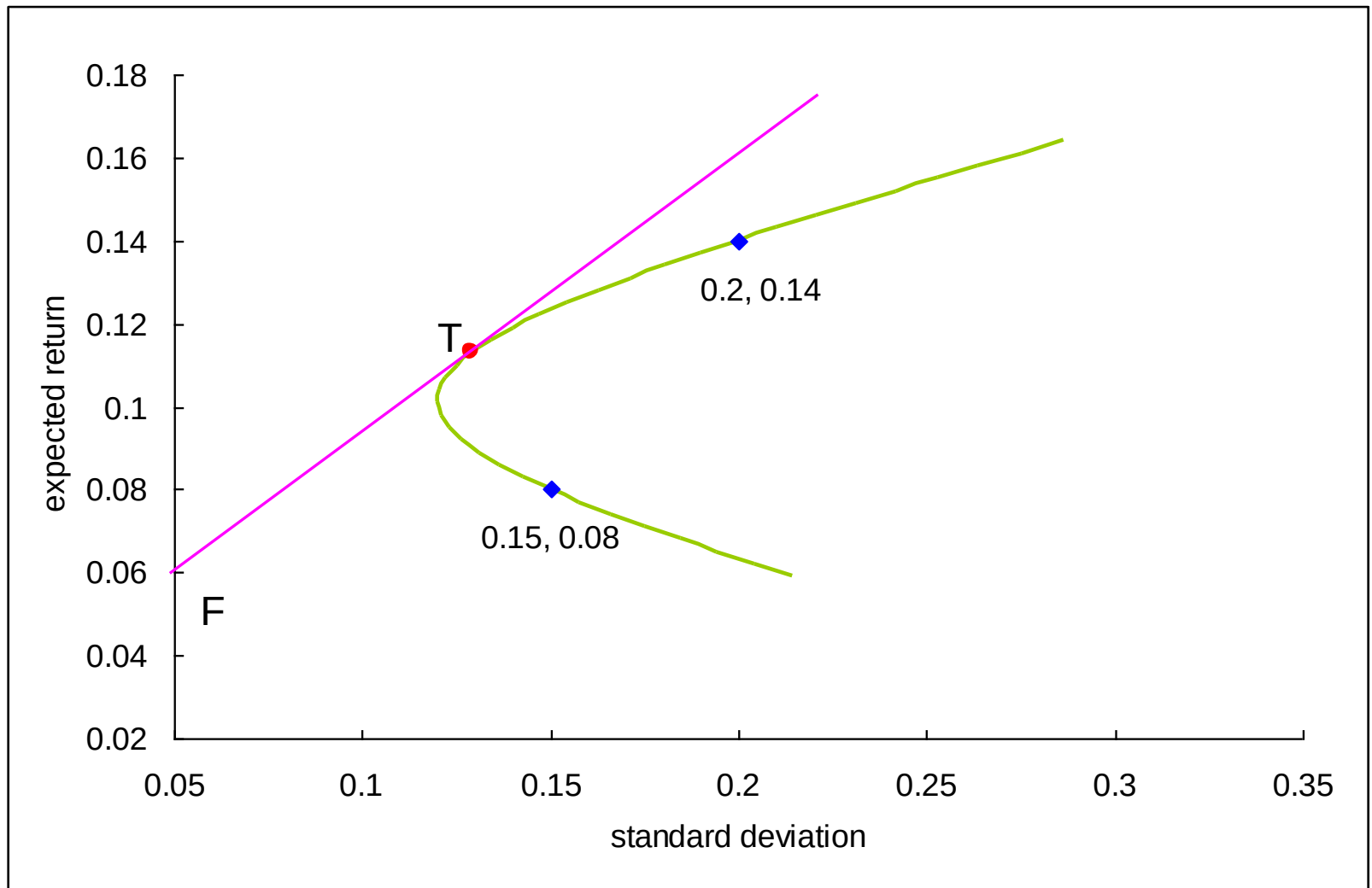


- The portfolio's mean and variance/standard deviation (shape) depend on the risky assets' means, variances and covariance.

The optimal combination of risky assets

- Consider the risk-reward combinations we can obtain by combining the riskless asset with risky asset 1 and risky asset 2.
- $r_f=0.06$.

Tangent portfolio



- Point T is the point of tangency between a straight line from point F drawn to the curve.
- We call this particular risky portfolio, which corresponds to the tangency point T, the optimal combination of risky assets.

IV. Achieving the Target Expected Return:

Example 1

- two common stocks with these statistics:
 - mean return 1 = 0.15
 - mean return 2 = 0.10
 - standard deviation 1 = 0.20
 - standard deviation 2 = 0.25
 - correlation of returns = 0.90
 - initial price 1 = \$57.25
 - Initial price 2 = \$72.625
- Riskless asset return = 5%
- What's the best return given a 10% standard deviation?

Share Prices



To find the tangent portfolio:

$$w_1^{\text{tan}} = \frac{(\mu_1 - r_f)\sigma_2^2 - (\mu_2 - r_f)\rho_{1,2}\sigma_1\sigma_2}{(\mu_2 - r_f)\sigma_1^2 - (\mu_1 - r_f + \mu_2 - r_f)\rho_{1,2}\sigma_1\sigma_2 + (\mu_1 - r_f)\sigma_2^2}$$

$$w_2^{\text{tan}} = 1 - w_1$$

$$w_1^{\text{tan}} = \frac{(0.10)*0.25^2 - (0.05)*0.90*0.20*0.25}{(0.05)*0.20^2 - (0.10 + 0.05)*0.90*0.20*0.25 + (0.10)*0.25^2}$$

$$w_1^{\text{tan}} = 2\frac{2}{3}$$

$$w_2^{\text{tan}} = -1\frac{2}{3}$$

- The tangent portfolio's mean and sd

$$\mu_{\text{tan}} = w_1^{\text{tan}} \mu_1 + w_2^{\text{tan}} \mu_2$$

$$\mu_{\text{tan}} = \frac{8}{3} 0.15 - \frac{5}{3} 0.10$$

$$\mu_{\text{tan}} = 0.2333$$

$$\sigma_{\text{tan}}^2 = \left(w_1^{\text{tan}}\right)^2 \sigma_1^2 + \left(w_2^{\text{tan}}\right)^2 \sigma_2^2 + 2w_1^{\text{tan}} w_2^{\text{tan}} \sigma_1 \sigma_2 \rho_{1,2}$$

$$\sigma_{\text{tan}}^2 = \left(\frac{8}{3}\right)^2 0.20^2 + \left(-\frac{5}{3}\right)^2 0.25^2 + 2\left(\frac{8}{3}\right)\left(-\frac{5}{3}\right) * 0.2 * 0.25 * 0.90$$

$$\sigma_{\text{tan}} = 0.2409$$

- The trade-off line

$$\mu = \frac{\mu_{\text{tan}} - r_f}{\sigma_{\text{tan}}} \sigma + r_f = \frac{0.2333 - 0.05}{0.2409} 0.10 + 0.05 = 0.1261$$

Achieving the Target Expected Return: weight

- Assume that the investment criterion is to generate a 30% return, what is the weight on the tangent portfolio?

$$\mu_{\text{criterion}} = \mu_{\text{tangent}} w_1 + r_f (1 - w_1)$$

$$w_1 = \frac{\mu_{\text{criterion}} - r_f}{\mu_{\text{tangent}} - r_f} = \frac{0.30 - 0.05}{0.2333 - 0.05} = 1.3636$$

Achieving the Target Expected Return: Volatility

- What's volatility associated with this portfolio?

$$\sigma = w_1 \sigma_{tangent} = 1.3636 * 0.2409 = 0.3285$$

- This is the volatility of the portfolio we seek

Achieving the Target Expected Return : Portfolio Weights

	COMPUTATION	WEIGHT
RISKLESS	-0.3636	-0.3636
ASSET 1	$1.3636 * 2.6667$	3.6363
ASSET 2	$1.3636 * (-1.6667)$	-2.2727
TOTAL		1.0000

V. Portfolios of many risky assets

- The same procedure as two risky assets (two-step). Portfolio's mean and variance:

$$\mu_P = \sum_{j=1}^n \omega_j \mu_j = (\omega_1, \omega_2, \dots, \omega_n) \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mathbf{M} \\ \mu_n \end{pmatrix}$$

$$\sigma_P^2 = \sum_{i=1}^n \sum_{j=1}^n \omega_i \omega_j \sigma_{ij} = (\omega_1, \omega_2, \dots, \omega_n) \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1n} \\ \mathbf{M} & \mathbf{O} & \mathbf{M} \\ \sigma_{n1} & \mathbf{L} & \sigma_{nn} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \mathbf{M} \\ \omega_n \end{pmatrix}$$

Portfolio Frontier

- Consider the expected return and risk of all feasible portfolios.
- Because of diversification, for **each level of expected return**, there will be a large number of portfolios, each with its own risk.
- **For each level of risk**, there will be a large number of portfolios, each with its own expected return.

- Given μ_p , choose $\omega_1, \omega_2, \dots, \omega_n$ to **minimize** σ_P^2 , subject to:

$$\mu_P = \sum_{j=1}^n \omega_j \mu_j, \quad \sum_{j=1}^n \omega_j = 1$$

- We call this curve the portfolio **frontier**.
- With many risky assets, the curve has the same properties as for just two risky assets.

- It is as if all assets are grouped into two composite assets – ‘composite asset’ = a portfolio of assets (a mutual fund)
- First **Mutual Fund Theorem** (or, Separation Theorem):

Assume:

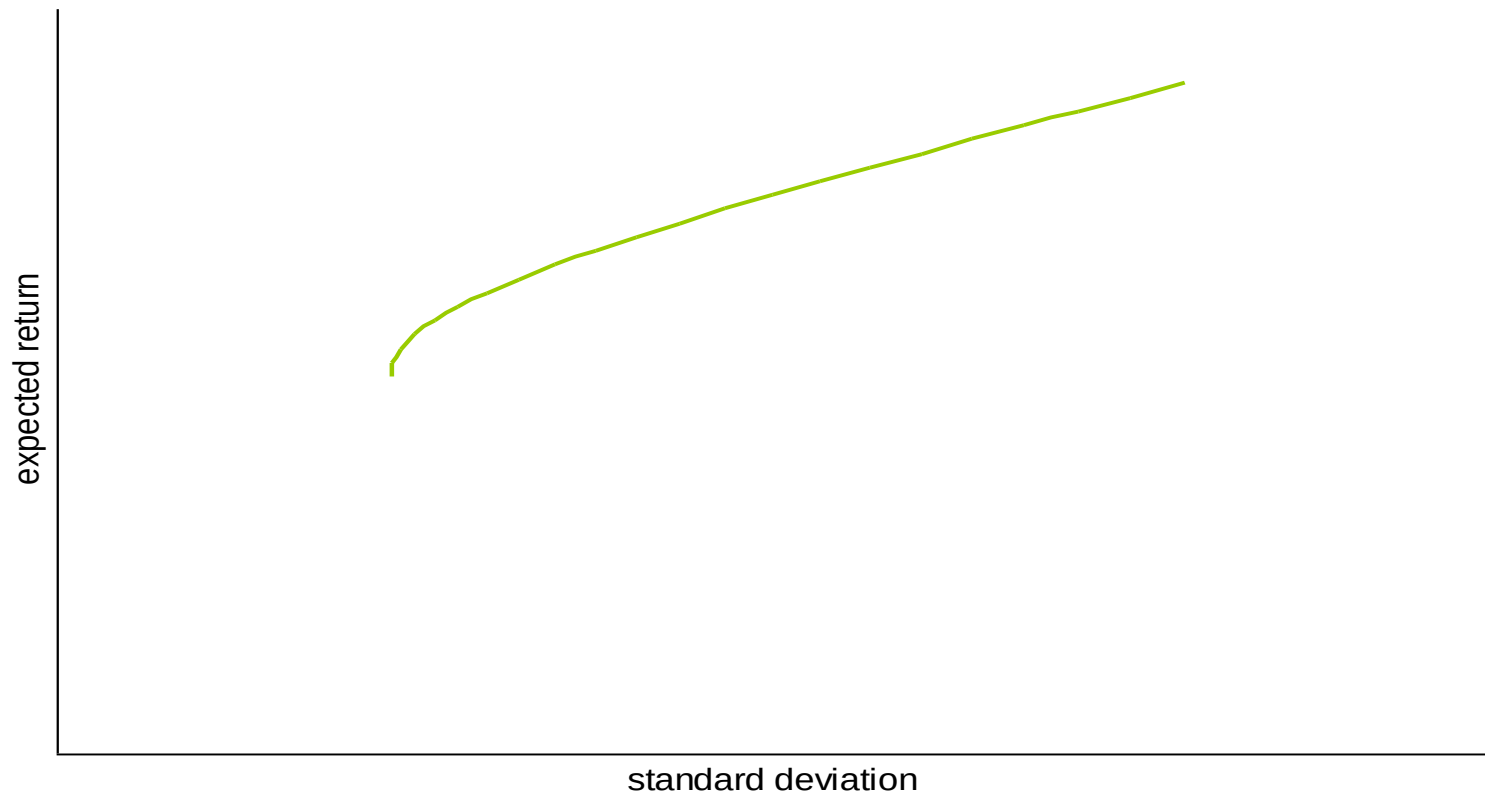
- (i) $n \geq 2$ assets;
- (ii) beliefs can be expressed in terms of means and variances. Then: there exist two mutual funds (composite assets) such that every optimal portfolio is some portfolio of the two mutual funds.

Markowitz Efficient Frontier

- For each level of risk there is a Markowitz efficient portfolio.
- The collection of all efficient portfolios is called the Markowitz efficient set of portfolios, or the Markowitz efficient frontier (MEF).

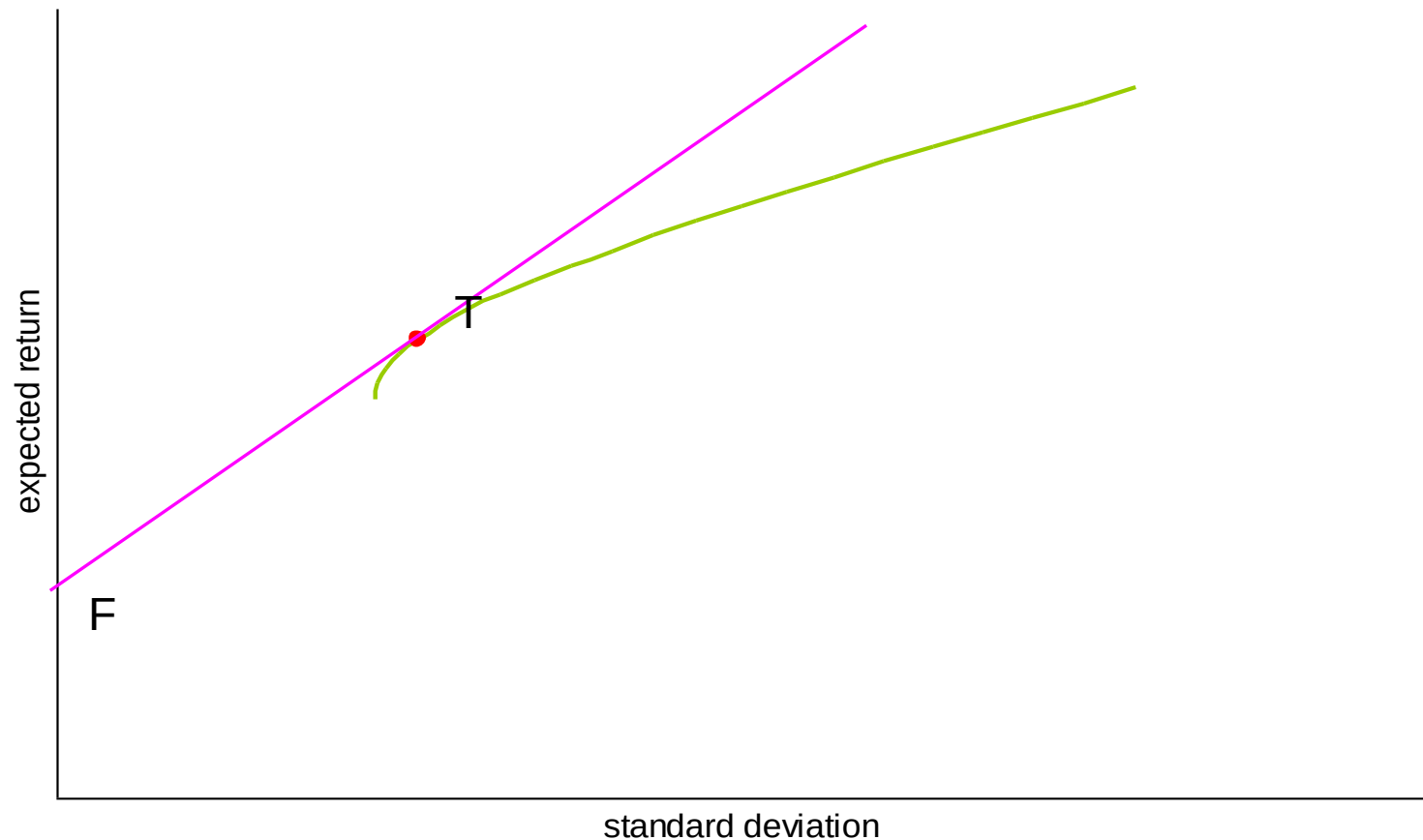
Efficient frontier

- Without riskless asset:



Efficient frontier

- With riskless asset



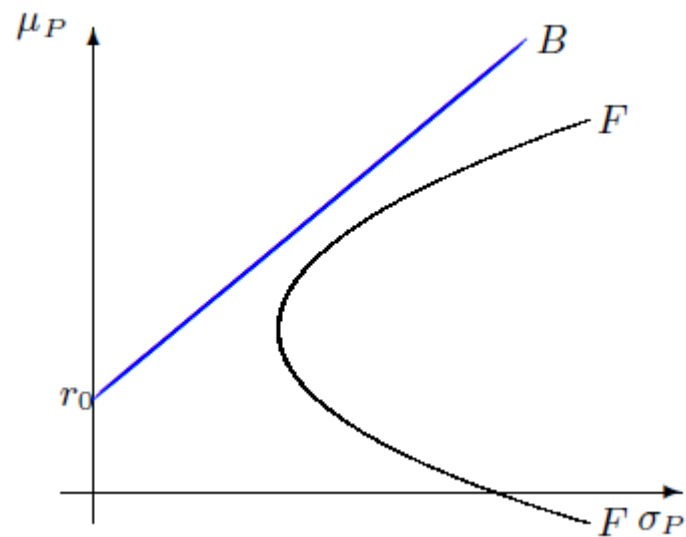
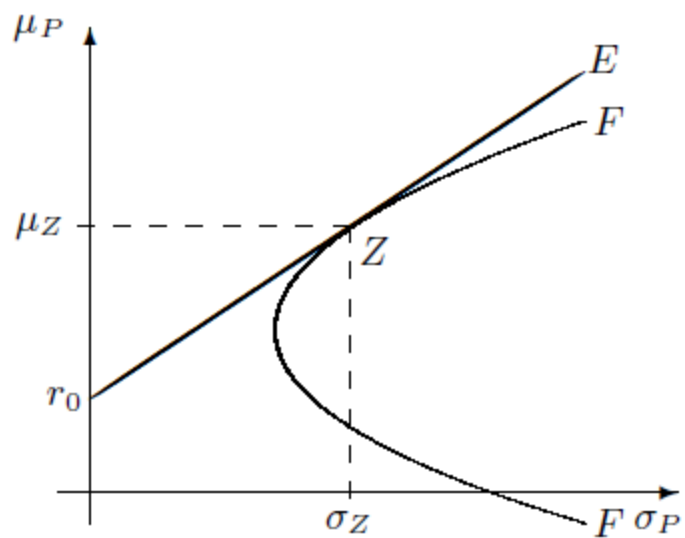
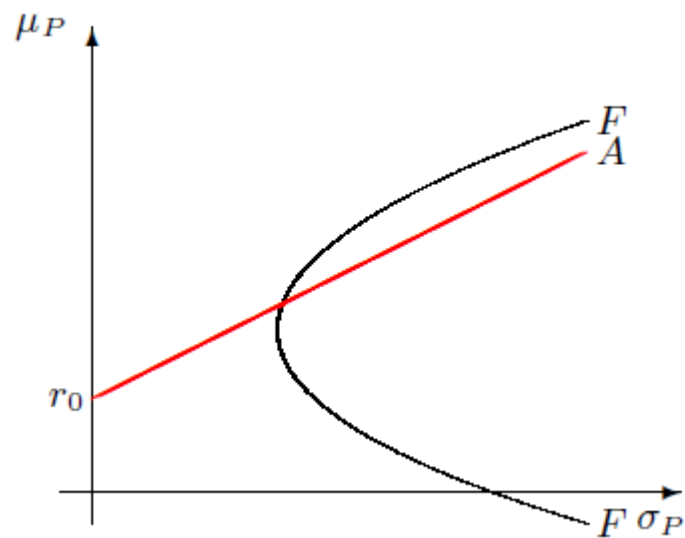
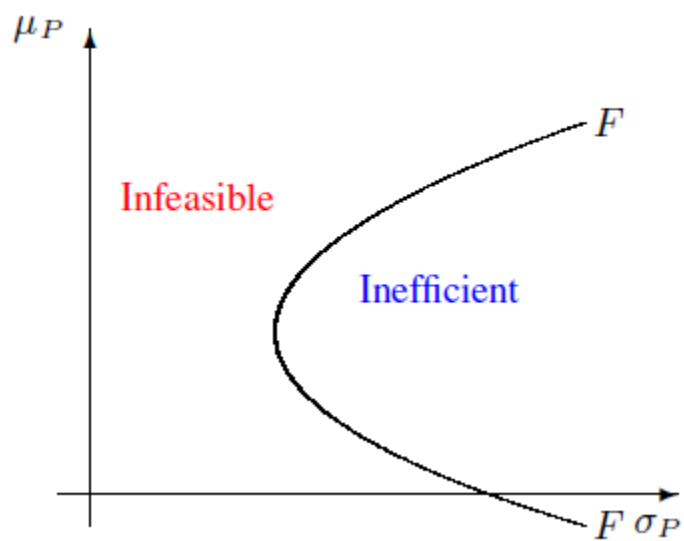
Feasible Set

➤ Feasible Set of Risky Assets

- An Efficient Portfolio is a combination of two different efficient portfolios (Two Fund Separation)

➤ Feasible Set with Riskless Asset

- Larger than the feasible set of risky assets
- the efficient frontier is a straight line that goes through the riskless asset and the tangency portfolio
- Need hold only the riskless asset and the tangency portfolio



VI. Optimal portfolio selection in MV

Recall:

- A mean-variance investor's optimal portfolio selection can be split logically into two steps:
- First, the **portfolio frontier**, comprising those portfolios for which σ_p^2 is minimized for each μ_p .
- Second, a choice is made from among the frontier portfolios so as to **maximize** the objective, U , in accordance with preference.

Example

- Suppose the investor has decided on the composition of the risky portfolio T . Now the concern is the proportion ω invested in the risky portfolio, and, $1-\omega$ invested in the risk-free asset. Assume:

$$E(r_T) = 15\%, \quad \sigma_T = 22\%, \quad r_f = 7\%$$

- The investor's utility function is:

$$U = E(r) - 0.5A\sigma^2$$

The combination portfolio's expected return and variance are:

$$E(r_p) = \omega E(r_T) + (1 - \omega)r_f = r_f + \omega(E(r_T) - r_f)$$

$$\sigma_p^2 = \omega^2 \sigma_T^2$$

The utility maximization problem:

$$\begin{aligned} \max_w U &= E(r_p) - 0.5A\sigma_p^2 \\ &= r_f + \omega(E(r_T) - r_f) - 0.5A\omega^2\sigma_T^2 \end{aligned}$$

How to solve the first order condition?

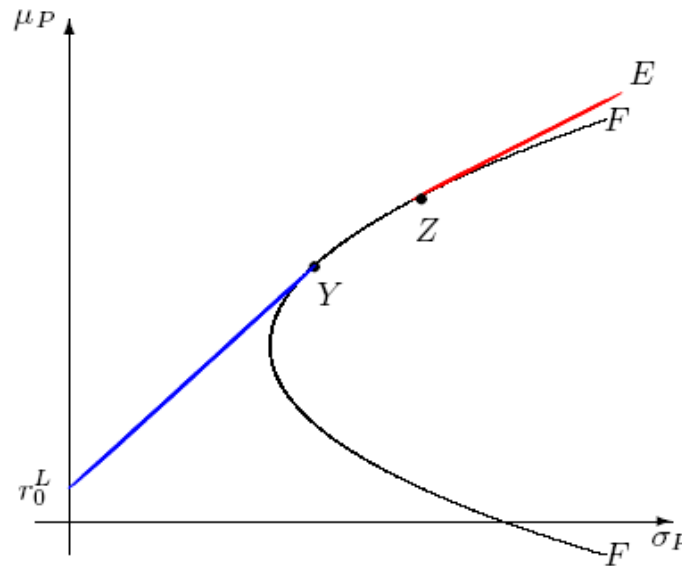
- Solving the first order condition:

$$\begin{aligned}\omega^* &= \frac{E(r_T) - r_f}{A\sigma_T^2} \\ &= \frac{0.15 - 0.07}{4 * 0.22^2} = .41\end{aligned}$$

$$E(r_p) = 0.07 + 0.41 * (0.15 - 0.07) = 10.28\%$$

$$\sigma_p = .41 * 0.22 = 9.02\%$$

Different borrowing and lending rates



- Lending at r_0^L allows portfolios along $r_0^L Y$
- Borrowing at r_0^B allows portfolios along ZE .
- Portfolio frontier has 3 segments:
 1. Lending: $r_0^L Y$
 2. Risky assets: $Y Z$
 3. Borrowing: ZE

The Sharpe ratio and risk-adjusted performance

- Modigliani and Modigliani (1997) define one measure of the risk-adjusted performance (RAP) based on Sharpe ratio:

$$RAP_j = r_f + \frac{(\mu_j - r_f)}{\sigma_j} \sigma_B$$

where σ_B is the standard deviation of the benchmark portfolio.

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