

Lecture 3 Decision Making under Uncertainty

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Key Points

- Uncertainty
- Expected utility
- Attitudes toward risk
- Measures of risk aversion
- Mean-variance analysis

I. Introduction to Uncertainty

- Frank Knight (1885-1962) distinguishes between the concepts of risk and uncertainty.
 - Risk: unknown events for which objective probabilities can be assigned.
 - Uncertainty: applies to events for which such probabilities cannot be assigned.
- A strict distinction between risk and uncertainty is not crucial, and the two words used interchangeably.

II. Expected utility

- Example 1: two distinct projects
 - (A): success — \$100 thousand (prob:48.4%)
fail — 0 (prob: 51.6%)
 - (B): success — \$64 thousand (prob: 60%)
fail — \$25 thousand (prob:40%)
- Choose which one?

- Project A:

Expected return $= 100 \times 48.4\% + 0 \times 51.6\% = 48.4$
thousand

- Project B:

Expected return $= 64 \times 60\% + 25 \times 40\% = 48.4$
thousand

- However, this does not mean that the two projects are equivalent to the investor at all.
- To enable him to judge which project is better to him, the expected utility theory introduces the concept of utility function.

- Utility Function:

Measures the **well-being** of investors (or more generically, an economic agent's well-being).

- an economic agent is presumably happier with larger wealth, assume that a utility function is increasing in wealth.

- Mathematically, utility function $u(.)$ has the property: $u'(>0)$

- Suppose in the above example:

$$u(x) = \sqrt{x}$$

$$Eu(x_A) = .484 \times 10 + .516 \times 0 = 4.84$$

$$Eu(x_B) = .6 \times 8 + .4 \times 5 = 6.8$$

Choose project 2.

Expected Utility

Two parts:

Probability and **utility function**

- Let X denote wealth (payoff/ return), when X is continuous:

$$Eu(X) = \int_x u(x) f(x) dx$$

$f(x)$ is the probability density function of X .

- When X is discrete variable:

$$Eu(X) = \sum_x \Pr\{X = x\} u(x)$$

The state-preference approach

- Three basic ingredient:
 1. **States**: Uncertainty is represented by **states of the world**: $S = \{s_1, s_2, \dots, s_l\}$
 2. **Actions**: Investors take actions – buy **portfolios of securities**. Action $a = \{x_1, x_2, \dots, x_n\}$
 3. **Consequences**: Actions have consequences – **outcomes** that differ across states $c = f(s_k, a)$
- Consequences are valued according to preferences.
- Preferences: $U = U(f(s_1, a), f(s_2, a), \dots, f(s_l, a))$.

III. Attitudes toward Risk

- Example 2: suppose the following two choices:

Invest		Not invest	
payoffs	Prob.	payoffs	Prob.
2	$\frac{1}{2}$	1	1
0	$\frac{1}{2}$		

- The investment cost is \$1, which one will you choose?

- Invest: the expected return $(1/2)*2+(1/2)*0=\$1$.
- Expected return=cost, we call this **fair game**.
- **Risk averse** investor will **reject** the fair game

$$U(1) > \frac{1}{2} U(2) + \frac{1}{2} U(0)$$

$$U(1)-U(0) > U(2)-U(1)$$

- The effect of decreasing utility of one unit is greater than that of increasing utility of one unit.

Three different attitudes toward risk:

- Risk averse $u''(.) < 0$,

Strictly concave utility function

- Risk neutral $u''(.) = 0$,

Linear utility function

- Risk loving $u''(.) > 0$,

Strictly convex utility function

- When $u''(.) < 0$, $Eu(X) < u(EX)$

When two projects have the same expected return, risk averse investor prefer the one with lower risk.

- When $u''(.) = 0$, $Eu(X) = u(EX)$

When two projects have the same expected return, risk neutral investor has no difference between the two.

- When $u''(.) > 0$, $Eu(X) > u(EX)$

When two projects have the same expected return, risk loving investor prefer the one with higher risk.

Example 3

- A random payoff: payoff = 100 with 50% prob., payoff = 10 with 50%. Now three agents who take different attitudes toward risk:
 - Agent A with utility function: $u_A(x) = \ln x$
 - Agent B with utility function $u_B(x) = 3x$
 - Agent C with utility function $u_C(x) = x^2$

Discuss: the relationship between $Eu(X)$? $u(EX)$

IV. Risk Premium

- Certainty equivalence

A payoff with certainty that is equivalent with a given random payoff.

$$u(\text{CE}) = Eu(X)$$

- Risk premium

The amount which one is ready to give up to attain a payoff with certainty.

$$r = EX - \text{CE}$$

Example 4

- Suppose wealth X follows the following distribution:
- $X=10$ (60%); 1 (40%)
- The agent with utility function: $u(X)=\ln X$

$$EX = ?$$

$$Eu(X) = ?$$

$$CE = ?$$

$$r = ?$$

V. Measures of Risk Aversion

Q: with the increasing in wealth, agent invests more in risky assets or not?

- Absolute risk aversion

$$ARA(x) = -\frac{u''(x)}{u'(x)}$$

- The curvature of the utility function matters.
- The degree of absolute risk aversion is higher as the curvature of the utility function is larger.

The degree of absolute risk aversion may change as the wealth level changes. With this respect, we classify the utility functions as follows:

- DARA (decreasing ARA): the degree of absolute risk aversion **diminishes** as the wealth level **increases**.
- CARA (constant ARA): ... **remains the same** as ...
- IARA (increasing ARA): ... **increases** as ...
- How about the investment in risky asset under different classification?

Example 5

- Logarithmic utility

$$U(x) = \ln x$$

- Power utility

$$u(x) = \frac{x^{1-c}}{1-c}, \text{ for } c > 0.$$

Measures of Risk Aversion

Q: with the increasing in wealth, agent invest more **proportion** in risky assets or not?

- Relative risk aversion

$$RRA(x) = -\frac{u''(x)}{u'(x)} x$$

- DRRA (decreasing RRA): the degree of relative risk aversion **diminishes** as the wealth level increases.
- CRRA (constant RRA): ...**remains the same**...
- IRRA (increasing RRA): ...**increases**...

- Logarithmic utility shows DARA and CRRA.
- Power utility shows DARA and CRRA.

Mean-variance analysis

- Quadratic utility

$$U(x) = x - bx^2$$

- The derivatives are

$$U'(x) = 1 - 2bx \quad (x < 1/2b)$$

$$U''(x) = -2b \quad (b > 0)$$

- The quadratic utility function has always had a special place in **mean variance** analysis because the assumption of a quadratic utility function leads to mean variance analysis being optimum.

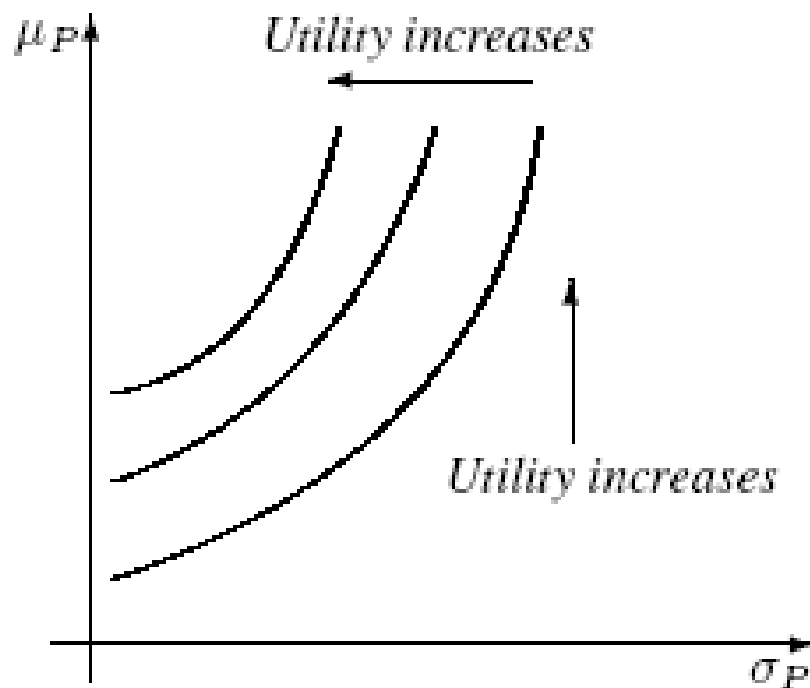
- Especially, the following function form has been used:

$$U = E(r) - 0.5A\sigma^2$$

where A is the risk aversion parameter.

- Portfolio receive **higher** utility scores for **higher expected returns** and **lower** scores for **higher volatility**.
- MV objective often expressed using rates of return, not wealth.
- Justifications for MV-mean variance:
 - a. **Quadratic** utility in the EUH-Expected Utility Hypothesis .
 - b. **Normal distributions** of returns in the EUH.
 - c. Directly, just that it seems plausible.

Indifference curves in μ_P , σ_P space



Example 6

- A portfolio has the expected return of 22% and standard deviation 34%. The risk-free rate in the market is 5%. Now there is a risk averse investor whose risk aversion coefficient is 3. Will she invest in this portfolio or not?
- And if her risk aversion coefficient decreases to 2, will she invest?

- For $A=3$,

Utility value is:

$$U = 0.22 - 0.5 * 3 * 0.34^2 = 4.66\% < 5\%,$$

One would reject the portfolio.

- When $A=2$,

$$U = 0.22 - 0.5 * 2 * 0.34^2 = 11.56\% > 5\%,$$

She would accept the portfolio.

VI. The empirical evidence about utility function

1. The assumption that investors prefer more to less is consistent with empirical evidence.
2. The assumption of risk aversion is also generally consistent with empirical evidence.
 - Research on ARA and RRA:
 - Blume and Friend: find the percentage invested in risky assets remains virtually unchanged.
 - Cohn, et al. : investors exhibit decreasing relative risk aversion.